1. Roots of a quadratic.

$\boxed{m = a^2 - b^2}$. If the roots are $r = a - b$ and s, then the sum of the roots is $r + s = 2a$; i.e., $a - b + s = 2a$. Thus $s = a + b$, and m is the product of the roots: $m = rs = (a - b)(a + b) = a^2 - b^2$.

2. Rational numbers.

It is the set of $\boxed{\text{all rational numbers}}$. For,

$$\frac{1}{2x + \sqrt{4x^2 + 1}} = \frac{2x - \sqrt{4x^2 + 1}}{4x^2 - (4x^2 + 1)} = -2x + \sqrt{4x^2 + 1},$$

and thus

$$2x + \sqrt{4x^2 + 1} - \frac{1}{2x + \sqrt{4x^2 + 1}} = 4x,$$

which is rational if and only if x is rational.

3. Last integer odd.

Suppose a and b are chosen, with $a \geq b$, and replaced by $a - b$. The sum of all the numbers in the list is reduced by $a + b - (a - b) = 2b$, so the parity is preserved. Initially the sum is $(2009)(2010)/2 = (2009)(1005)$, which is odd. Then the sum is odd after each move, so when a single element remains it is odd.

SECOND SOLUTION

Initially there are an odd number of odd numbers on the list. At each step, if a and b are not both odd, the number of odd numbers on the list will be unchanged, and if a and b are both odd, the number of odd numbers on the list will be reduced by two, so there will always be an odd number of odd numbers on the lsit. Thus when only one number remaims, it must be odd.

4. Smallest C.

It is $\boxed{C = 64}$. We note that $f(x) > 0$ for all $x > 0$, $f(x)$ grows without bound as $x \to 0$ and as $x \to \infty$, so $f(x)$ has a positive minimum value on $(0, \infty)$, and $f'(x)$ must be zero where this minumum occurs. $f'(x) = 6x - 3Cx^{-4}$, which is 0 only when $2x^5 = C$. Thus $f(x)$ takes

its minimum value at $x_0 = (C/2)^{1/5}$, and we want $f(x_0) \geq 20$.

$$\begin{aligned} f(x_0) &= 3x_0^2 + Cx_0^{-3} \\ &= 3\left(\frac{C}{2}\right)^{\frac{2}{5}} + 2\left(\frac{C}{2}\right)\left(\frac{C}{2}\right)^{-\frac{3}{5}} \\ &= 5\left(\frac{C}{2}\right)^{\frac{2}{5}}. \end{aligned}$$

Thus $f(x) \geq 20$ for all $x > 0$ provided that $5(\frac{C}{2})^{\frac{2}{5}} \geq 20$; i.e., $(\frac{C}{2}) \geq 4^{\frac{5}{2}} = 32$; i.e., $C \geq 64$. Thus, the smallest such C is 64.

5. Counting ordered quadruples of integers.

There are $\boxed{3634}$ such quadruples. If $b = a + k$, then $c = d - k$, and

$$ad + 2009 = bc = (a + k)(d - k) = ad + kd - ka - k^2,$$

so $k(d - a - k) = 2009 = 7^2 \cdot 41$. Because $d - k = c > b = a + k$, we must have $d > a + 2k$, and therefore $d - a - k > k$. Thus the possible values for k are 1, 7 and 41. With $k = 1$ we have $d - a - k = 2009$, and thus $d > 2009$, so there are no solutions with $k = 1$.

With $k = 7$ we have $d - a - k = 2009/7 = 287$ and $d - a = 287 + k = 294$, so $d = 294 + a$ and $c = d - k = 287 + a$, giving us quadruples $(a, a + 7, a + 287, a + 294)$, with the condition that $a + 294 \leq 2009$, so $1 \leq a \leq 1715$.

With $k = 41$ we have $d - a - k = 49$, and $d - a = 49 + 41 = 90$, giving quadruples $(a, a + 41, a + 49, a + 90)$, with $a + 90 \leq 2009$, so $1 \leq a \leq 1919$. In all, then, there are $1715 + 1919 = 3634$ such quadruples.

6. Bases for F^3 over F.

There are $\boxed{248{,}000}$ different bases. A basis is a linearly independent unordered set of three vectors from V. We will first count the linearly independent ordered sets of three vectors. There are $5^3 = 125$ vectors in V. To form a linearly independent set, first choose any nonzero vector v_1. There are 124 from which to choose. For a second vector we may choose any which is not a scalar multiple av_1 of v_1. There are 5 scalar multiples of v_1, so we have $125 - 5 = 120$ choices for v_2 making $\{v_1, v_2\}$ a linearly independent set. For the third vector we can choose any vector v_3 which is not a linear combination $av_1 + bv_2$ of the first two. There are 25 such linear combinations to avoid, so the third vector may be any of the remaining 100 vectors. This gives us $124 \cdot 120 \cdot 100$ linearly independent ordered triples of vectors. Each unordered set of three has been counted $3! = 6$ times, so there are $124 \cdot 120 \cdot 100/6 = 248,000$ unordered linearly independent sets of three vectors.

7. No consecutive heads.

The probability is $\boxed{89/512}$. Let h_n be the number of sequences of n tosses in which no HH occurs and the last toss is H. Let t_n be the number of sequences of n tosses in which no HH occurs and the last toss is T. Then

$$h_{n+1} = t_n \quad \text{and} \quad t_{n+1} = h_n + t_n. \tag{1}$$

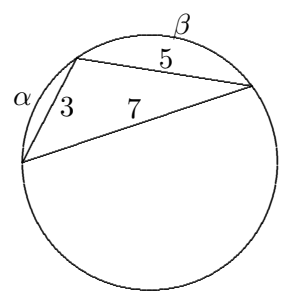
Now starting from $h_2 = 1$ and $t_2 = 2$ we can construct the following table:

n	2	3	4	5	6	7	8	9
h_n	1	2	3	5	8	13	21	34
t_n	2	3	5	8	13	21	34	55

Thus the number of admissible sequences of length nine is $h_9 + t_9 = 89$, the total number of possible outcomes of 9 tosses is $2^9 = 512$, and the desired probability is $89/512$.
(Note that from (1) we have $t_{n+2} = h_{n+1} + t_{n+1} = t_n + t_{n+1}$, so both the sequences $\{t_n\}$ and $\{h_n = t_{n-1}\}$ are Fibonacci sequences.)

8. Chords of length 3,5,7.

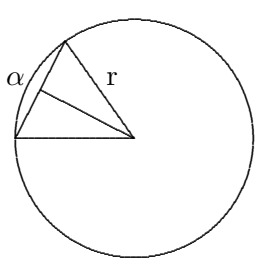
We will show that $\boxed{\cos\alpha = \dfrac{71}{98}}$. Place the chords of lengths 3 and 5 end to end as in the figure at the right. Then a chord of length 7 completes a triangle as shown. The angle opposite the side of length 3 is $\alpha/2$, and by the law of cosines,



$$\cos\frac{\alpha}{2} = \frac{5^2 + 7^2 - 3^2}{2 \cdot 5 \cdot 7} = \frac{13}{14}.$$

Then $\cos\alpha = 2\cos^2\dfrac{\alpha}{2} - 1 = 2\left(\dfrac{169}{196}\right) - 1 = \dfrac{71}{98}$.

SECOND SOLUTION



If r is the radius of the circle, then we have $\sin(\alpha/2) = 3/2r$, $\sin(\beta/2) = 5/2r$), and

$$\frac{7}{2r} = \sin\frac{\alpha+\beta}{2} = \sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \sin\frac{\beta}{2}\cos\frac{\alpha}{2} = \frac{3}{2r}\sqrt{1 - \frac{25}{4r^2}} + \frac{5}{2r}\sqrt{1 - \frac{9}{4r^2}}.$$

Multiply through by $4r$ and put $q = 1/r^2$ and the equation becomes

$$14 = 3\sqrt{4 - 25q} + 5\sqrt{4 - 9q};$$

$$196 = 9(4-25q)+25(4-9q)+30\sqrt{(4-25q)(4-9q)} = 136-450q+30\sqrt{16-136q+225q^2}.$$

Thus $(2+15q)^2 = 16-136q+225q^2$, and this simplifies further to

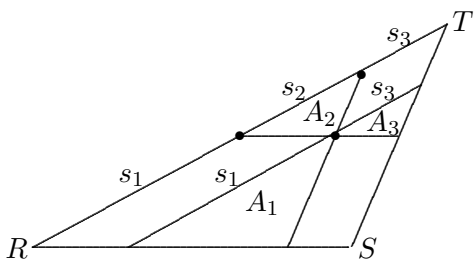
$$\frac{1}{r^2} = q = \frac{12}{96},$$

so $1/r = \sqrt{3}/7$. Then

$$\sin\frac{\alpha}{2} = \frac{3}{2}\frac{\sqrt{3}}{7} \quad \text{and} \quad \cos\frac{\alpha}{2} = \sqrt{1-\frac{27}{196}} = \frac{13}{14},$$

so $\cos\alpha = 71/98$ as in the first solution.

9. Square roots of area ratios.



The three smaller triangles are all similar to RST, so their areas are proportional to the squares of the corresponding sides. Let $s = RT$ and s_1, s_2, s_3 be the lengths of the corresponding sides of the inner triangles (numbered in the same order as A_1, A_2, A_3). Then $s = s_1 + s_2 + s_3$, and

$$\frac{A_1}{A} = \frac{s_1^2}{s^2}, \qquad \frac{A_2}{A} = \frac{s_2^2}{s^2}, \qquad \text{and} \qquad \frac{A_3}{A} = \frac{s_3^2}{s^2}.$$

Thus

$$\sqrt{\frac{A_1}{A}} + \sqrt{\frac{A_2}{A}} + \sqrt{\frac{A_3}{A}} = \frac{s_1}{s} + \frac{s_2}{s} + \frac{s_3}{s} = \frac{s_1+s_2+s_3}{s} = 1.$$

10. Final digits of 2009^n.

We seek n so that $2009^n = 10^{2010}m + 1$ for some integer m; equivalently, that

$$2009^n - 1 = 10^{2010}m. \tag{1}$$

Consider the 10^{2010} integers 2009, 2009^2, 2009^3, $\ldots$, $2009^{10^{2010}}$, and let us look at their remainders modulo 10^{2010}. The remainder 0 cannot occur because no power of 2009 ends in 0. Thus some two remainders are the same; i.e., there exist integers r and s with $0 < r < s \le 10^{2010}$ such that

$$2009^r \equiv 2009^s \pmod{10^{2010}}.$$

Then

$$2009^r(2009^{s-r}-1) = 2009^s - 2009^r = 10^{2010}k$$

for some integer k. There are no factors of 10 in 2009^r, so

$$2009^{s-r} - 1 = 10^{2010}m$$

for some integer m. Thus, (1) is satisfied with $n = s - r$.

SECOND SOLUTION

Here is a proof by induction that for every integer $m \geq 0$ there is a positive integer n such that 2009^n ends in 1 preceded by m zeros. With $m = 0$ we have 2009^2 ending in 1 (preceded by no zeros). Suppose that 2009^n ends with 1 preceded by $m-1$ zeros; say $2009^n = 1+q10^m$. Then

$$2009^{10n} = (1 + q10^m)^{10} = 1 + 10q10^m + \cdots = 1 + r10^{m+1}$$

for some integer r, so ends in 1 preceded by m zeros.

THIRD SOLUTION

Since $\gcd(2009, 10^{2010}) = 1$, 2009 is an element of the multiplicative group $\{k < 10^{2010} : \gcd(k, 10) = 1\}$, where multiplication is done modulo 10^{2010}. Hence, for some n, $2009^n \equiv 1 \pmod{10^{2010}}$. This n does the job.

FOURTH SOLUTION

Euler's generalization of Fermat's Little Theorem says that if integers a and m are relatively prime, then $a^{\phi(m)} \equiv 1 \pmod{m}$, where ϕ is Euler's totient function. Well, $a = 2009$ and $m = 10^{2010}$ are relatively prime, so

$$2009^{\phi(m)} \equiv 1 \pmod{10^{2010}}.$$