Each problem number is followed by an 11-tuple $(a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$, where a_k is the number of teams that scored k points on the problem.

1. Rational and irrational numbers. (36,1,3,0,1,1,1,1,1,2,27)

From the fact that

$$\frac{w+a}{w+b} = 1 + \frac{a-b}{w+b}$$

it is clear that if $\frac{w+a}{w+b}$ is rational, so is $\frac{a-b}{w+b}$. Thus we may write

$$\frac{a-b}{w+b} = \frac{p}{q},$$

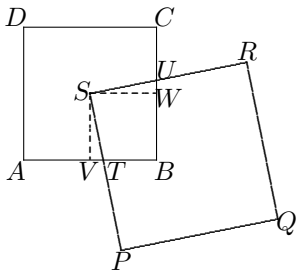
where p and q are integers and $q \neq 0$. It follows that

$$pw = qa - qb - pb.$$

In this equation the right member is rational, so pw is rational. But with p an integer and w irrational, pw is rational only if $p = 0$, which implies $a - b = 0$.

2. Overlapping squares. (66,0,0,0,0,0,2,0,0,1,5)

The area is $\boxed{25}$. Drop perpendiculars from S to V on side AB and from S to W on side BC. Then triangles SVT and SWU are congruent, and the area of quadrilateral $STBU$ is equal to the area of the square $SVBW$ of side 5, so is 25.



3. Powers of a matrix. (17,1,1,0,0,0,0,0,0,2,53)

We are given that the characteristic polynomial of A is $x^2 - x - 1$. By the Cayley-Hamilton Theorem, $A^2 - A - I = 0$, so $A^2 = A + I$, and (1) is satisfied for $n = 2$. Suppose that (1) holds for $n = k$. Then

$$\begin{aligned} A^{k+1} = A(A^k) &= A(F_{k-1}A + F_{k-2}I) \\ &= F_{k-1}A^2 + F_{k-2}A \\ &= F_{k-1}(A + I) + F_{k-2}A \\ &= (F_{k-1} + F_{k-2})A + F_{k-1}I \\ &= F_kA + F_{k-1}I, \end{aligned}$$

so by induction (1) holds for all $n \geq 2$.

4. Minimum value. (10,0,0,7,11,5,0,2,6,6,27)

The minimum value is $\boxed{-4}$. Keeping in mind that $y \neq 0$ in the domain of f, one calculates directly that

$$\begin{aligned} f(z) &= \frac{5x^4y - 10x^2y^3 + y^5}{y^5} = \frac{5x^4 - 10x^2y^2 + y^4}{y^4} \\ &= \frac{5x^4 - 10x^2y^2 + 5y^4 - 4y^4}{y^4} = \frac{5(x^2 - y^2)^2}{y^4} - 4 \geq -4, \end{aligned}$$

and at each point where $x = y$ we have $f(z) = -4$, so -4 is the minimum value.

5. 2008 points on a disc. (28,0,0,0,1,1,1,1,10,1,31)

Draw a circle of radius 2 about each of the 2008 points. The total area of these circles is $2008 \cdot 4\pi$, so collectively they cover at most an area of 8032π within the larger circle. The area of the circle of radius 90 is 8100π, so we may choose a point within the large circle that is outside all of the 2008 circles of radius 2. This point has the desired property.

6. Sum of two certain terms. (30,3,2,2,2,2,2,1,1,3,26)

We'll show that $\boxed{x_{63} + x_{61} = 2008}$. Consider the equations

$$\begin{aligned} x_4 &= x_3 - x_2 + x_1 + 4 \\ x_5 &= x_4 - x_3 + x_2 + 5 \\ &\vdots \\ x_n &= x_{n-1} - x_{n-2} + x_{n-3} + n. \end{aligned}$$

Summing these equations and denoting $x_1 + x_2 + \cdots + x_n$ by S_n we have

$$\begin{aligned} S_n - x_1 - x_2 - x_3 = (S_n - x_1 - x_2 - x_n) - (S_n - x_1 - x_{n-1} - x_n) \\ + (S_n - x_{n-2} - x_{n-1} - x_n) + \frac{(n+4)(n-3)}{2}, \end{aligned}$$

which simplifies to

$$x_n + x_{n-2} = x_3 + x_1 + \frac{(n+4)(n-3)}{2} = -2 + \frac{(n+4)(n-3)}{2}.$$

Then putting $n = 63$ we get $x_{63} + x_{61} = -2 + (67)(60)/2 = 2008$.

7. Winning the fourth game . (32,0,0,1,0,4,3,0,8,2,24)

The probability that Adolf wins game four is $\boxed{\frac{40}{81}}$. On each game the probability that the first player wins is

$$\frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\cdots=\frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}=\frac{2}{3}.$$

Let P_k be the probability that Adolf wins the k-th game. Then

$$P_k=\frac{1}{3}P_{k-1}+\frac{2}{3}(1-P_{k-1})=\frac{2}{3}-\frac{1}{3}P_{k-1}.$$

From this we calculate successively:

$$\begin{aligned}
P_1&=\frac{2}{3};\\
P_2&=\frac{2}{3}-\frac{1}{3}\cdot\frac{2}{3}=\frac{4}{9};\\
P_3&=\frac{2}{3}-\frac{1}{3}\cdot\frac{4}{9}=\frac{14}{27};\\
P_4&=\frac{2}{3}-\frac{1}{3}\cdot\frac{14}{27}=\frac{40}{81}.
\end{aligned}$$

(One could also draw the tree of possible outcomes and add the probabilities along the eight paths leading to a win for Adolf in game four.)

8. Limit of a sequence (31,0,3,8,1,1,0,0,2,1,27)

The limit is $\boxed{mk(k+1)/2}$. For each integer r with $1\le r\le k$, consider

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{(n+r)^m}{n^{m-1}}-n\right)&=\lim_{n\to\infty}\left(\frac{(1+r/n)^m}{1/n}-n\right)\\
&=\lim_{n\to\infty}\frac{(1+r/n)^m-1}{1/n}\\
&=r\lim_{n\to\infty}\frac{(1+r/n)^m-1^m}{r/n}\\
&=rf'(1),
\end{aligned}$$

where $f(x)=x^m$. Then $f'(x)=mx^{m-1}$, and $rf'(1)=rm$. Adding these for $1\le r\le k$, we have

$$\lim_{n\to\infty}\left(\frac{(n+1)^m+(n+2)^m+\cdots+(n+k)^m}{n^{m-1}}-kn\right)=m+2m+\cdots+km=\frac{mk(k+1)}{2}.$$

9. An inequality. (17,0,3,0,0,0,9,2,1,1,41)

We show that $\boxed{\sum_{k=1}^{2007}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}=2008-\frac{1}{2008}}$. We have

$$\begin{aligned}1+\frac{1}{k^2}+\frac{1}{(k+1)^2}&=\frac{(k+1)^2k^2+(k+1)^2+k^2}{k^2(k+1)^2}\\&=\frac{k^4+2k^3+3k^2+2k+1}{k^2(k+1)^2}=\frac{(k^2+k+1)^2}{k^2(k+1)^2},\end{aligned}$$

so we may rewrite the sum

$$\begin{aligned}\sum_{k=1}^{2007}\frac{k^2+k+1}{k^2+k}&=\sum_{k=1}^{2007}\Big(1+\frac{1}{k(k+1)}\Big)=2007+\sum_{k=1}^{2007}\Big(\frac{1}{k}-\frac{1}{k+1}\Big)\\&=2007+\Big(1-\frac{1}{2008}\Big)=2008-\frac{1}{2008}.\end{aligned}$$

10. Equal sums of fractions. (3,0,1,1,1,1,0,1,1,6,59)

Denote the left member of the equality to be proved by L and the right member by R. We will add

$$S=\Big(\frac{2007}{2}+\frac{2006}{3}+\frac{2005}{4}+\cdots+\frac{1005}{1004}\Big)+\Big(\frac{1004}{1005}+\frac{1003}{1006}+\cdots+\frac{2}{2007}+\frac{1}{2008}\Big)$$

to both members, and show that $L+S=R+S$.

$$\begin{aligned}L+S&=\Big(\frac{1}{1005}+\frac{1004}{1005}\Big)+\Big(\frac{3}{1006}+\frac{1003}{1006}\Big)+\cdots+\Big(\frac{2007}{2008}+\frac{1}{2008}\Big)\\&\qquad+\Big(\frac{2007}{2}+\frac{2006}{3}+\cdots+\frac{1005}{1004}\Big)\qquad(1)\\&=1004+\Big(\frac{2007}{2}+\frac{2006}{3}+\cdots+\frac{1005}{1004}\Big).\end{aligned}$$

$$R+S=2\Big(\frac{2007}{2}+\frac{2005}{4}+\cdots+\frac{1}{2008}\Big)=2007+\frac{2005}{2}+\frac{2003}{3}+\cdots+\frac{3}{1003}+\frac{1}{1004}.\qquad(2)$$

Subtracting (1) from (2) we obtain

$$\begin{aligned}R-L&=(2007-1004)+\Big(\frac{2005-2007}{2}\Big)+\Big(\frac{2003-2006}{3}\Big)+\cdots\\&\qquad+\Big(\frac{3-1006}{1003}\Big)+\Big(\frac{1-1005}{1004}\Big)\\&=1003+1003(-1)=0,\end{aligned}$$

as there are 1003 fractions of the form $\Big(\frac{2009-2k}{k}-\frac{2009-k}{k}\Big)=-1$.