TWELFTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 15, 2008, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit for significant but incomplete work. For full credit, answers must be fully justified. But in some cases this may simply mean showing all work and reasoning. Have fun!

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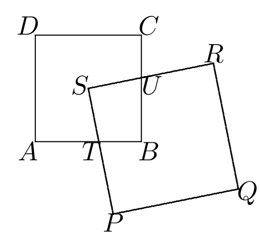
1. Rational and irrational numbers.

Given that a and b are rational numbers and that w is irrational, show that

$$\text{if } \frac{w+a}{w+b} \text{ is rational, then } a=b.$$

2. Overlapping squares.

In the figure at the right, $ABCD$ is a square of side 10 and $PQRS$ is a square of side 12. The vertex S is at the center of square $ABCD$, side SP intersects side AB at T, dividing AB in the ratio 3:2, and side SR intersects side BC at U. Find the area of the quadrilateral $STBU$, and justify your answer.



3. Powers of a matrix.

The Fibonacci sequence $\{F_n\}$ is defined by $F_0 = F_1 = 1$ and for $n > 1$, $F_n = F_{n-1} + F_{n-2}$. Suppose that A is a square matrix with characteristic polynomial $\det(xI - A) = x^2 - x - 1$, where $\det M$ denotes the determinant of the matrix M, and I is the identity matrix of the same dimensions as A. Prove that

$$A^n = F_{n-1}A + F_{n-2}I \quad \text{for all } n \geq 2. \tag{1}$$

4. Minimum value.

For complex numbers $z = x + yi$ with x and y real and $y \neq 0$, let $f(z) = \dfrac{\Im(z^5)}{(\Im z)^5}$, where $\Im z$ denotes the imaginary part of z (i.e., $\Im(x+yi) = y$). Find, with proof, the minimum value of $f(z)$ over its domain.

5. 2008 points in a disc.

Within a circle of radius 90 in the plane there are 2008 points selected. Prove that there is a new point within the circle whose distance from each of the 2008 selected points is at least 2.

6. Sum of two certain terms.

A sequence is defined recursively by $x_1 = 2007$, $x_2 = 2008$, $x_3 = -2009$, and for $n > 3$, $x_n = x_{n-1} - x_{n-2} + x_{n-3} + n$. Find $x_{63} + x_{61}$.

7. Winning the fourth game.

Adolf and Bertha play a game in which they take turns tossing a fair coin, and the first to toss heads wins. Adolf tosses first in the first game, and thereafter the loser of a game goes first in the next game. If they play four games, what is the probability that Adolf wins the fourth game?

8. Limit of a sequence.

For given positive integers m and k, evaluate

$$\lim_{n\to\infty}\left(\frac{(n+1)^m + (n+2)^m + \cdots + (n+k)^m}{n^{m-1}} - kn\right).$$

Justify your answer.

9. An inequality.

Show that

$$\sum_{k=1}^{2007}\sqrt{1 + \frac{1}{k^2} + \frac{1}{(k+1)^2}} < 2008.$$

10. Equal sums of fractions.

Prove that

$$\frac{1}{1005} + \frac{3}{1006} + \frac{5}{1007} + \cdots + \frac{2007}{2008} = \frac{2007}{2} - \frac{2006}{3} + \frac{2005}{4} - \cdots - \frac{2}{2007} + \frac{1}{2008}.$$