ELEVENTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 10, 2007, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit for significant but incomplete work. For full credit, answers must be fully justified. But in some cases this may simply mean showing all work and reasoning. Have fun!

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1. Logarithmic sum.

The function f is defined on the positive integers by

$$f(n) = \begin{cases} \log_{32} n & \text{if } \log_{32} n \text{ is rational} \\ 1 & \text{otherwise.} \end{cases}$$

Evaluate

$$\sum_{n=1}^{2007} f(n).$$

2. A magic square.

In a magic square, all row sums, column sums, and the two diagonal sums are equal. Three entries are given in a 3×3 square at the right. Fill in the remaining entries with any real numbers whatever to make it a magic square.

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3. The coefficient of x^2.

Let $P_0(x) = x^3 + ax^2 - 1018x + 2007$, and for integers $n \geq 1$, let $P_n(x) = P_{n-1}(x - n)$. If the coefficient of x in $P_{10}(x)$ is 2007, what is the coefficient of x^2 in $P_{10}(x)$?

4. Integral of fractional part.

For real x, let $\langle x \rangle$ denote the fractional part of x. Thus, $\langle x \rangle = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Evaluate

$$\int_{-1}^{1} \langle x^2 + 2x - 3 \rangle dx.$$

5. Difference of squares.

Find all ordered pairs (a, b) of positive integers such that

$$a^2 - b^2 = 2007.$$

6. Sum of square roots.

Find all pairs of positive integers (x, y) with $x \leq y$ such that $\sqrt{x} + \sqrt{y} = \sqrt{2007}$.

7. Difference of square roots.

Is there an integer N satisfying the following equation?

$$(\sqrt{2007} - \sqrt{2006})^{2008} = \sqrt{N} - \sqrt{N-1}.$$

8. Group product.

Elements $a_1, a_2, \ldots, a_n$ are selected sequentially at random, with replacement (so they are not necessarily distinct) from a multiplicative group G of order n, with identity 1. Show that there exist integers r and s, with $1 \leq r \leq s \leq n$, such that

$$\prod_{k=r}^{s} a_k = 1.$$

9. Limit of a sequence of integrals.

Evaluate

$$\lim_{n \to \infty} \int_{n}^{2n} \frac{n^3 x dx}{x^5 + 1}.$$

10. Lattice points on a curve.

Show that the only lattice points (x, y) on the curve

$$y^2 = x^4 + x^3 + x^2 + x + 1$$

are $(-1, \pm 1), (0, \pm 1)$ and $(3, \pm 11)$. (Lattice points are points with integer coordinates.)