Each problem number is followed by an 11-tuple $(a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$, where a_k is the number of teams that scored k points on the problem.

1. GP identity. (34,1,0,0,0,0,1,0,1,4,17)

If r is the common ratio then $b = ar$ and $c = ar^2$, and thus $ac = a^2r^2 = b^2$. Then

$$a^2b^2c^2 = a^2(ac)c^2 = a^3c^3 = b^6,$$

and

$$\begin{aligned} a^2b^2c^2(a^{-3} + b^{-3} + c^{-3}) &= a^3c^3(a^{-3} + b^{-3} + c^{-3}) \\ &= c^3 + a^3c^3b^{-3} + a^3 \\ &= c^3 + b^3 + a^3. \end{aligned}$$

This is obviously equivalent to the desired equation.

2. An integral. (31,1,0,2,0,1,1,0,0,0,22)

The value is $\boxed{\ln \frac{(10)(101)(1002)(2009)}{(11)(102)(1003)}}$. We can calculate it as follows:

$$\begin{aligned} \int_1^{2006} \frac{dx}{x + \lfloor \log_{10} x \rfloor} &= \int_1^{10} \frac{dx}{x} + \int_{10}^{100} \frac{dx}{x+1} + \int_{100}^{1000} \frac{dx}{x+2} + \int_{1000}^{2006} \frac{dx}{x+3} \\ &= \ln 10 + \ln \frac{101}{11} + \ln \frac{1002}{102} + \ln \frac{2009}{1003} \\ &= \ln \frac{(10)(101)(1002)(2009)}{(11)(102)(1003)}. \end{aligned}$$

3. A simple product. (35,0,0,0,1,3,0,1,0,2,16)

The product is $\boxed{\frac{2007}{2 \cdot 2006} = \frac{2007}{4012}}$. Let

$$\Pi_n := \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{n^2}\right).$$

Examination of some cases for small n suggests the following formula, which we will prove by induction for every integer $n \geq 2$:

$$\Pi_n = \frac{n+1}{2n}.$$

With $n = 2$ we have

$$1 - \frac{1}{2^2} = \frac{3}{4} = \frac{n+1}{2n}.$$

Suppose that

$$\Pi_k = \frac{k+1}{2k}.$$

Then

$$\Pi_{k+1} = \Pi_k\left(1 - \frac{1}{(k+1)^2}\right) = \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$
$$= \frac{(k+1)(k+2)k}{2k(k+1)^2} = \frac{k+2}{2(k+1)},$$

and by induction the claim is established.

4. Sums of powers of 3. (9,2,2,4,8,0,9,3,6,3,12)

There are $\boxed{120}$. The powers of 3 which may be used are 1, 3, 9, 27, 81, 243 and 729. The sum of all seven of these is smaller than 2006, so every sum of two or more of them is in the range 1 to 2006. Of the $2^7 = 128$ subsets of this set of seven numbers, one is empty and seven are singletons. Each of the remaining 120 subsets produces one such integer. It remains to show that different subsets yield different integers. Note that

$$1 + 3 + \cdots + 3^k = \frac{3^{k+1}-1}{3-1} < 3^{k+1}.$$

Suppose that

$$3^{a_1} + 3^{a_2} + \cdots + 3^{a_r} = 3^{b_1} + 3^{b_2} + \cdots + 3^{b_s},$$

where $a_1 > a_2 > \cdots > a_r$ and $b_1 > b_2 > \cdots > b_s$. I claim that then $r = s$ and $a_k = b_k$ for $k = 1, 2, \ldots, r$. If not, let m be the least integer for which $a_m \neq b_m$. Then

$$3^{a_m} + 3^{a_{m+1}} + \cdots + 3^{a_r} = 3^{b_m} + 3^{b_{m+1}} + \cdots + 3^{b_s}.$$

Now, if $a_m > b_m$, then $3^{a_m} > 3^{b_m} + \cdots + 3^{b_s}$, contradicting the assumed equality. Similarly $a_m < b_m$ is impossible, and the claim is established.

SECOND SOLUTION

The problem is equivalent to counting the positive integers which in base 3 have seven or fewer digits, where each digit is 0 or 1, and at least two 1s are present. There are 2^7 seven-digit strings of 0s and 1s, of which 7 have exactly one 1 and 1 has no 1s. Thus the count is $2^7 - 7 - 1 = 120$.

5. An integer combination. (35,0,0,1,0,0,0,0,0,0,22)

Such integers exist. One such triple is $\boxed{(1, -11, 10)}$. Rewrite the equation

$$20(100a + 10b + c) + 6(a + b + c) = 0.$$

Now set $100a+10b+c = 0$ and $a+b+c = 0$ and look for integer solutions. We get $a = c/10$ and $b = -11c/10$. Take $c = 10$ and get the solution $(a, b, c) = (1, -11, 10)$.

SECOND SOLUTION

Another way to find such a triple is by the Euclidean algorithm. It is equivalent that $1003a + 103b + 13c = 0$. Now, 1003 and 103 are relatively prime, so there are integers m, n such that $1003m + 103n = 1$. The Euclidean algorithm produces $m = -42$, $n = 409$. Then we may take $a = -13m,\ b = -13n,\ c = 1$, giving us the triple $\boxed{(a, b, c) = (546, -5317, 1)}$.

6. A cube and its reciprocal. (21,0,1,11,2,1,1,0,1,1,19)

We show that $\boxed{r^3 + \frac{1}{r^3} = 5778}$. From $\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}} = 3$ we have upon cubing

$$r + 3\sqrt[3]{r} + \frac{3}{\sqrt[3]{r}} + \frac{1}{r} = 27,$$

so

$$r + \frac{1}{r} = 27 - 3\left(\sqrt[3]{r} + \frac{1}{\sqrt[3]{r}}\right) = 27 - 9 = 18.$$

Cubing once more we obtain

$$r^3 + 3\left(r + \frac{1}{r}\right) + \frac{1}{r^3} = 18^3,$$

and so

$$r^3 + \frac{1}{r^3} = 18^3 - 3 \cdot 18 = 5778.$$

7. The largest term. (22,0,0,3,1,0,1,0,0,0,31)

The largest term is $\boxed{T_{182}}$. To see this, look at the ratio

$$\begin{aligned}\frac{T_{k+1}}{T_k} &= \frac{\binom{2006}{k+1}(0.1)^{k+1}}{\binom{2006}{k}(0.1)^k} \\ &= \frac{2006!(0.1)^{k+1}}{(k+1)!(2006-k-1)!} \cdot \frac{k!(2006-k)!}{2006!(0.1)^k} \\ &= \frac{(2006-k)(0.1)}{k+1}.\end{aligned}$$

We see that $T_{k+1} > T_k$ iff $(2006-k)(0.1) > k+1$; i.e., iff

$$k < \frac{(2006)(0.1)-1}{1.1} = \frac{1996}{11} = 181\frac{5}{11}.$$

Thus, for $k \leq 181$, $T_{k+1} > T_k$, and for $k \geq 182$, $T_{k+1} < T_k$. The maximum term is T_{182}.

8. Integral roots. (8,0,2,0,0,4,11,0,10,4,19)

There are four such numbers: $\boxed{b = -16, 0, 20 \text{ or } 36}$. Here is a proof. Let m and n be integer roots of the quadratic equation $x^2 + bx + 5b = 0$. Then $m + n = -b$ and $mn = 5b$, so b is an integer, and $mn + 5(m+n) = 0$. It follows that

$$(m+5)(n+5) = mn + 5(m+n) + 25 = 25.$$

Because $m+5$ is an integer it must be $\pm 1, \pm 5$ or ± 25. Because of the symmetry in m and n, we may assume without loss of generality that $m \leq n$, so we have the possible pairs $(m+5, n+5) = (-25, -1), (-5, -5), (1, 25)$ and $(5, 5)$, and the corresponding pairs $(m, n) = (-30, -6), (-10, -10), (-4, 20)$ and $(0, 0)$. The corresponding values of $b = -(m+n)$ are $36, 20, -16$ and 0.

9. A special rational approximation. (7,0,1,0,0,0,0,1,1,0,48)

Put $m_0 = 1$ and $n_0 = 1000$, and let $\delta = \dfrac{m_0^2}{n_0^3} = \dfrac{1}{10^9} < 10^{-8}$. Let k be the integer such that

$$k\delta \le \sqrt{2006} < (k+1)\delta.$$

Then $0 \le \sqrt{2006} - k\delta < \delta$, so $|\sqrt{2006} - k\delta| < \delta < 10^{-8}$. Let $m = k^2 m_0$ and $n = kn_0$. Then

$$\begin{aligned}\left|\frac{m^2}{n^3} - \sqrt{2006}\right| &= \left|\sqrt{2006} - \frac{k^4 m_0^2}{k^3 n_0^3}\right| = \left|\sqrt{2006} - k\frac{m_0^2}{n_0^3}\right| \\ &= |\sqrt{2006} - k\delta| < 10^{-8}.\end{aligned}$$

SECOND SOLUTION

(This was essentially the solution offered by the Bemidji State Univ. team WINNERS and by the Winona State Univ. team Alpha.)

The rational numbers are dense in the reals. Let p and q be integers such that

$$|p/q - \sqrt{2006}| < 10^{-8}.$$

Let $m = p^2 q$ and $n = pq$. Then $m^2/n^3 = p/q$, so this m and n do the job.

10. Infinitely many square terms. (13,1,0,1,0,0,0,0,0,0,43)

The second term is $2025 = 45^2$. To show there are infinitely many squares it suffices to show that given any square term $19 + 2006r = m^2$, there is a larger square in the sequence. Let k be an integer such that $2006k > 2m$, and take $n = 2006k - m$. Then $n > m$, and

$$n^2 - m^2 = (n-m)(n+m) = (n-m)(2006k),$$

so $n^2 = m^2 + 2006k(n-m)$ is also in the progression.

SECOND SOLUTION (Courtesy of Carleton "Sophomores" and UMN-TC "Maroon")

The second term is 45^2. Note that $(a + 2006)^2 - a^2 \equiv 0 \pmod{2006}$. Therefore if a^2 is in the sequence, so is $(a + 2006)^2$.