Solutions, 2003 NCS/MAA TEAM COMPETITION

Each problem number is followed by an 11-tuple $(a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$, where a_k is the number of teams that scored k points on the problem.

1. Doubling the area.(48,6,3,4,0,0,2,0,0,1,1)

The function is $r = (\sqrt{2}+1)t$. From the given information we have that

$$\pi(r+t)^2 = 2\pi r^2.$$

Thus

$$r^2 + 2rt + t^2 = 2r^2;$$
$$r^2 - 2rt + t^2 = 2t^2,$$

so

$$|r - t| = \sqrt{2}t.$$

Now, $t < r$, because increasing the radius by r would quadruple the area: $\pi(2r)^2 = 4\pi r^2$. Thus $|r-t| = r - t$, and we have $r - t = \sqrt{2}t$; $r = (\sqrt{2}+1)t$.

2. Simultaneous equations.(11,0,2,2,0,2,1,3,2,10,32)

The solutions are $(x, y) = (1, 2),\ (2, 1),\ (-3 + i\sqrt{2}, -3 - i\sqrt{2})$ or $(-3 - i\sqrt{2}, -3 + i\sqrt{2})$. One way to solve the system: Add twice the second equation to the first to get

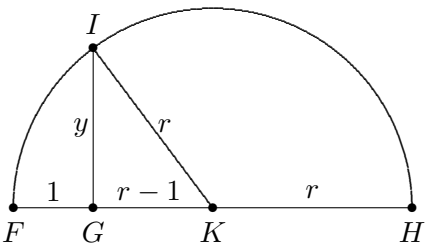
$$(x+y)^2 + 3(x+y) = 18.$$

Transpose the 18 and factor:

$$(x+y-3)(x+y+6) = 0.$$

Thus $x + y = 3$ or -6. With $x + y = 3$ we have from the second equation $xy = 2$ and the solutions are $(x, y) = (2, 1)$ or $(1, 2)$. With $x + y = -6$ we have $xy = 11$, and the solutions are $(x, y) = (-3 + i\sqrt{2}, -3 - i\sqrt{2})$ or $(-3 - i\sqrt{2}, -3 + i\sqrt{2})$.

3. Geometric construction of square root.(44,1,0,1,0,0,0,1,2,0,16)



Let r be the radius of the circle, the y be the length of GI. Then $GK = r - 1$ and $KI = r$, and by the Pythagorean Theorem, $(r-1)^2 + y^2 = r^2$. Expanding and simplifying we obtain $y^2 = 2r - 1 = r + (r-1) = GH$.■

4. Half heads.(13,0,0,2,1,0,0,9,8,5,27)

P_n is always larger. It suffices to show that P_n is a decreasing function of n, and for this it suffices to show that $P_{n+1} < P_n$ for all n. We have

$$P_n = \frac{\binom{2n}{n}}{2^{2n}} \quad \text{and} \quad P_{n+1} = \frac{\binom{2n+2}{n+1}}{2^{2n+2}},$$

so

$$\begin{aligned}\frac{P_n}{P_{n+1}} &= \frac{\binom{2n}{n}}{2^{2n}} \cdot \frac{2^{2n+2}}{\binom{2n+2}{n+1}} = \frac{(2n)!}{n!n!} \cdot \frac{4(n+1)!(n+1)!}{(2n+2)!} \\ &= \frac{4(n+1)^2}{(2n+1)(2n+2)} = \frac{2n+2}{2n+1} > 1.\end{aligned}$$

Thus $P_n > P_{n+1}$ for all n. It follows that $P_n > P_m$ whenever $m > n$. In particular, $P_n > P_{2n}$.

5. Powers of a matrix.(19,2,7,4,2,5,2,2,1,1,20)

(a) Examination of

$$M = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 7 & -12 \\ 3 & -5 \end{pmatrix}$$

suggests the formula

$$M^n = \begin{pmatrix} 1+2n & -4n \\ n & 1-2n \end{pmatrix}, \tag{1}$$

which we will prove by induction to be correct for all positive integers n. We see that $n = 1$ in (1) gives M. Suppose that (1) is valid for $n = k$. Then

$$\begin{aligned}M^{k+1} = M^k M &= \begin{pmatrix} 1+2k & -4k \\ k & 1-2k \end{pmatrix} \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{pmatrix} \\ &= \begin{pmatrix} 3+2k & -4-4k \\ k+1 & -1-2k \end{pmatrix} \\ &= \begin{pmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{pmatrix},\end{aligned}$$

showing that (1) is correct for $n = k+1$. Thus (1) is proved by induction.

(b) Putting $n = -\frac{1}{2}$ in (1) gives the matrix

$$\begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix}.$$

This is $M^{-\frac{1}{2}}$ in the sense that its square is M^{-1}, the multiplicative inverse of M:

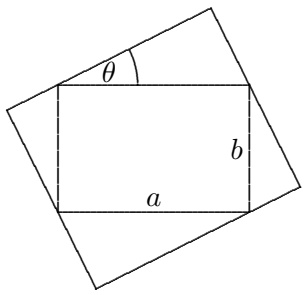
$$\begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix},$$

and this is M^{-1}, because

$$\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}.$$

6. Circumscribed rectangle.(8,0,0,1,4,5,1,2,1,9,34)

Let θ be the angle in the range $0 \le \theta \le \frac{\pi}{2}$ formed by side a of the given rectangle and one of the sides of the circumscribing rectangle. The outer rectangle is partitioned into five parts, consisting of the inner rectangle and two pairs of congruent right triangles, from which we see that



$$\begin{aligned} m^2 &= ab + (a\sin\theta)(a\cos\theta) + (b\sin\theta)(b\cos\theta) \\ &= ab + \frac{1}{2}(a^2+b^2)\sin 2\theta. \end{aligned}$$

The extreme values of this function for $0 \le \theta \le \frac{\pi}{2}$ are seen by inspection to be ab and $\frac{1}{2}(a+b)^2$, so m takes all values in the range

$$\sqrt{ab} \le m \le \frac{a+b}{\sqrt{2}}.$$

7. Lattice point solutions.(7,0,1,0,3,3,1,2,7,17,24)

There are six such pairs: $(2005, 2003^2+2002)$, $(2003, -2003^2+2002)$, $(4007, 4005)$, $(1, -1)$, $(2003^2+2004, 2003)$, $(-2003^2+2004, 2001)$. We have

$$\begin{aligned} (x-2004)(y-2002) &= xy - 2004y - 2002x + 2004\cdot 2002 \\ &= 2004\cdot 2002 + 1 = 2003^2. \end{aligned}$$

Now 2003 is prime, so $x - 2004$ must be one of ± 1, ± 2003, $\pm 2003^2$. These yield the six solutions given above.

8. Series of cubes.(1,0,0,0,0,0,0,0,0,0,64)

No. Here is one counterexample. Begin with the convergent alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}} = 1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \cdots.$$

Now split each of the positive terms into two equal parts, to get the series

$$\frac{1}{2} + \frac{1}{2} - \frac{1}{\sqrt[3]{2}} + \frac{1}{2\sqrt[3]{3}} + \frac{1}{2\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \cdots.$$

The pair of successive terms

$$\frac{1}{\sqrt[3]{2n-1}} - \frac{1}{\sqrt[3]{2n}}$$

in the alternating series becomes

$$\frac{1}{2\sqrt[3]{2n-1}} + \frac{1}{2\sqrt[3]{2n-1}} - \frac{1}{\sqrt[3]{2n}}. \tag{1}$$

It is easy to see that this series still converges: The partial sums here are either partial sums in the original series, or differ from such a partial sum by $1/2\sqrt[3]{2n+1}$, which tends to 0.

The series $\sum_{k=1}^{\infty} a_k^3$, however, diverges. The cubes of the three successive terms (1) yield the sum

$$\frac{1}{8(2n-1)} + \frac{1}{8(2n-1)} - \frac{1}{2n} = \frac{-1}{4}\left(\frac{3n-2}{2n^2-n}\right),$$

from which one sees that the partial sums tend to $-\infty$.

9. A null sequence of integrals.(3,0,1,6,2,0,5,4,6,0,38)

First integrate by parts, using $u = f(x),\ du = f'(x)\,dx,\ dv = \sin nx\,dx,\ v = -\frac{1}{n}\cos nx$. This give us

$$\begin{aligned}\int_a^b f(x)\sin nx\,dx &= \left[-\frac{1}{n}f(x)\cos nx\right]_a^b + \frac{1}{n}\int_a^b f'(x)\cos nx\ dx \\ &= \frac{1}{n}[f(a)\cos na - f(b)\cos nb] + \frac{1}{n}\int_a^b f'(x)\cos nx\,dx.\end{aligned}$$

Now

$$\left|\frac{1}{n}[f(a)\cos na - f(b)\cos nb]\right| \le \frac{1}{n}(|f(a)\cos na| + |f(b)\cos nb|) \le \frac{1}{n}(|f(a)| + |f(b)|) \to 0$$

as $n \to \infty$. For the other term, note that $f'(x)$, being continuous on $[a, b]$, is bounded there.

Thus there is a real number M such that $|f'(x)| \leq M$ on $[a, b]$, and

$$\begin{aligned}\left|\frac{1}{n}\int_a^b f'(x)\cos nx\,dx\right| &\leq \frac{1}{n}\int_a^b |f'(x)||\cos nx|dx \\ &\leq \frac{1}{n}\int_a^b M\,dx = \frac{M(b-a)}{n} \to 0\end{aligned}$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty}\int_a^b f(x)\sin nx\,dx = 0.$$

10. Differential-integral equation.(1,0,0,0,0,2,0,0,0,0,62)

Let $g(x) = \min\{x, 1-x\}$, and note that g is continuous and $\int_0^1 g(x)dx = \frac{1}{4}$. Let $M = 4\int_0^1 |f(x)|dx$. Then

$$\int_0^1 (|f(x)| - Mg(x))dx = \frac{M}{4} - \frac{M}{4} = 0,$$

so by continuity of the integrand and the Intermediate Value Theorem, there is a point c in $(0, 1)$ where $|f(c)| = Mg(c)$. By the Mean Value Theorem,

$$|f(c)| = |f(c) - f(0)| = |(c-0)f'(u_1)| = c|f'(u_1)|$$

for some u_1 in $(0, c)$, and

$$|f(c)| = |f(c) - f(1)| = |(c-1)f'(u_2)| = (1-c)|f'(u_2)|$$

for some u_2 in $(c, 1)$. Thus

$$|f'(u_1)| = \frac{|f(c)|}{c} = \frac{Mg(c)}{c}$$

and

$$|f'(u_2)| = \frac{|f(c)|}{(1-c)} = \frac{Mg(c)}{(1-c)}.$$

If $0 < c \leq \frac{1}{2}$, then $g(c) = c$ and $|f'(u_1)| = M$. If $\frac{1}{2} < c < 1$, then $g(c) = 1-c$ and $|f'(u_2)| = M$. Thus in every case we have a point u in $(0, 1)$ for which

$$|f'(u)| = M = 4\int_0^1 |f(x)|dx.$$