# SEVENTH ANNUAL
# NORTH CENTRAL SECTION MAA
# TEAM CONTEST

November 15, 2003

9:00 a.m. to 12:00 noon

**To the team members:** These problems are meant to be fun as well as challenging. If you can work all ten of them in the allotted time, super! But if not, you are likely to have lots of good company. Partial credit will be given for significant progress or for significant partial solutions, but a thorough job on a few will usually be better than some exploratory work on many. Each problem counts 10 points.

**NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.**

Each team may submit one solution to each problem. Think of your solution as an essay; a logical argument which makes clear why your answer to the question is correct, or why the assertion whose proof is called for in the problem is true.

**PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER.** Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

**1. Doubling the area.**

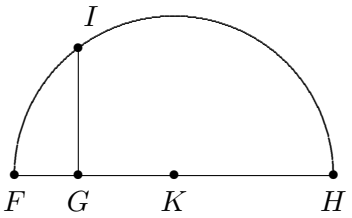
The positive numbers $r$ and $t$ are related by the fact that if the radius $r$ of a circle is increased by $t$, the area is doubled. Express $r$ as a function of $t$.

**2. Simultaneous equations.**

Find all solutions $(x, y)$ in the complex number system of the system of equations:

$$\begin{aligned} x^2 + y^2 + x + y &= 8 \\ xy + x + y &= 5 \end{aligned}$$

**3. Geometric construction of square root.**




In "La Géométrie", Descartes gives the following geometric construction of a square root: "If the square root of $GH$ is desired, I add, along the same straight line, $FG$ equal to unity; then bisecting $FH$ at $K$, I describe the circle $FIH$ about $K$ as center, and draw from $G$ a perpendicular and extend it to $I$, and $GI$ is the required root." The figure at the left accompanies Descartes' prescription. Assuming, as the figure does, that $GH > 1$, prove that the length $GI$ is the required root.

**4. Half heads.**

Let $n$ be a positive integer. Let $P_n$ be the probability that in $2n$ tosses of a fair coin exactly $n$ heads occur, and $P_{2n}$ the probability that in $4n$ tosses of a fair coin exactly $2n$ heads occur. Which is larger, $P_n$ or $P_{2n}$? Does the answer depend on $n$, and if so, how? Defend your answer.

**5. Powers of a matrix.**

Let $M$ be the $2 \times 2$ matrix

$$M = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}.$$

(a) Find a formula for $M^n$, and prove that it is valid for all positive integers $n$.

(b) Put $n = -\frac{1}{2}$ in your formula in (a), and interpret the resulting matrix. Is it in some sense $M^{-\frac{1}{2}}$?

**6. Circumscribed rectangle.**

A rectangle with sides $a$ and $b$ is circumscribed by another rectangle of area $m^2$. Determine all possible values of $m$ in terms of $a$ and $b$.

**7. Lattice point solutions.**

Find all pairs $(x, y)$ of integers such that

$$1 + 2002x + 2004y = xy.$$

(We note that 2003 is a prime.)

**8. Series of cubes.**

Given that the real number series $\sum_{k=1}^{\infty} a_k$ converges, does it follow that $\sum_{k=1}^{\infty} a_k^3$ converges? Prove it, or give a counterexample.

**9. A null sequence of integrals.**

Let $a$ and $b$ be real numbers, with $-\infty < a < b < \infty$, and let $f$ be a function in the class $C^1$ (i.e., the derivative $f'$ exists and is continuous) on the interval $[a, b]$. Show that

$$\lim_{n\to\infty} \int_a^b f(x) \sin nx \, dx = 0.$$

**10. Differential-integral equation.**

Let $f$ be a differentiable function on $[0, 1]$, with $f(0) = f(1) = 0$. Prove that there is a point $u$ in $(0, 1)$ where

$$|f'(u)| = 4 \int_0^1 |f(x)| dx.$$