Solutions, 2002 NCS/MAA TEAM COMPETITION

Each problem number is followed by an 11-tuple $(a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$, where a_k is the number of teams that scored k points on the problem.

1. Even periodic. (43,6,1,3,0,2,0,0,0,3,12)

The answer is $4-\pi$. For, $f(-\pi) = f(2-\pi) = f(4-\pi) = 4-\pi$, because $0 < 4-\pi < 1$. (That f is even is actually irrelevant.)

2. An arithmetic progression. (35,0,2,0,1,0,0,1,3,4,24)

The $(p+q)$th term is 0. Let the first term be a and the common difference d. Then

$$a + (p-1)d = q$$

and

$$a + (q-1)d = p.$$

Subtract the second equation from the first to get

$$(p-q)d = q - p.$$

We are given that $p - q \neq 0$, so $d = -1$, and the $(p+q)$th term is

$$\begin{aligned} a + (p+q-1)d &= [a + (p-1)d] + qd \\ &= q + q(-1) \\ &= 0. \end{aligned}$$

3. This year's term. (51,0,0,0,3,6,1,1,0,0,8)

It is 169. We have $f_1(101) = (1+0+1)^2 = 2^2 = 4$; $f_2(101) = 4^2 = 16$; $f_3(101) = (1+6)^2 = 7^2 = 49$; $f_4(101) = (4+9)^2 = 13^2 = 169$; $f_5(101) = 16^2 = 256$; $f_6(101) = 13^2 = f_4(101)$. Then for $n \geq 6$, $f_n(101) = f_{n-2}(101)$, so $f_{2k}(101) = 169$ and $f_{2k-1}(101) = 256$. In particular, $f_{2002}(101) = f_4(101) = 169$.

4. A definite integral. (11,0,1,0,1,1,1,0,2,18,35) The value is $e^{\frac{\sqrt{\pi}}{2}} - 1$. Write the integrand $e^{-x^2}e^{f(x)}$ and make the substitution

$$u = f(x),\ du = f'(x)dx = e^{-x^2}dx.$$

When $x = 0,\ u = f(0) = 0$, and as $x \to \infty,\ u \to \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$. Thus

$$\int_0^\infty e^{-x^2}e^{f(x)}dx = \int_0^{\frac{\sqrt{\pi}}{2}} e^u du = e^{\frac{\sqrt{\pi}}{2}} - 1.$$

5. Average distance 1/2. (11,0,0,0,2,0,0,1,15,6,35)

Define the function f on $[0,1]$ by

$$f(x) = \frac{1}{n}\sum_{i=1}^{n} |x - a_i|.$$

We wish to show that $f(x) = 1/2$ for some x in $[0,1]$. At $x = 0$ we have

$$f(0) = \frac{1}{n}\sum_{i=1}^{n} |-a_i| = \frac{1}{n}\sum_{i=1}^{n} a_i.$$

At $x = 1$ we have

$$\begin{aligned} f(1) &= \frac{1}{n}\sum_{i=1}^{n} |1 - a_i| \\ &= \frac{1}{n}\sum_{i=1}^{n}(1 - a_i) \\ &= \frac{1}{n}\left(n - \sum_{i=1}^{n} a_i\right) \\ &= 1 - f(0), \end{aligned}$$

so that

$$f(0) + f(1) = 1.$$

Thus, either $f(0) = f(1) = 1/2$, or the values $f(0)$ and $f(1)$ are on opposite sides of 1/2. Because f is continuous, in the latter case the intermediate value theorem implies that there is an x in $(0,1)$ with $f(x) = 1/2$. In either case we have the desired result.

6. Solve for n. (28,5,5,3,3,5,1,2,0,0,18)

The only solution is $n = 95$. The condition may be interpreted as requiring that the average of the numbers on the left side be 2. For $1 \le k \le 15$, $\lfloor \sqrt[4]{k} \rfloor = 1$. For $16 \le k \le 80$, $\lfloor \sqrt[4]{k} \rfloor = 2$. For $81 \le k \le 255$, $\lfloor \sqrt[4]{k} \rfloor = 3$, and we need exactly 15 of these size 3 terms to make the average term in our sum equal to 2. Thus, taking $n = 95$, we have

$$\sum_{k=1}^{95} \lfloor \sqrt[4]{k} \rfloor = 15 \cdot 1 + 65 \cdot 2 + 15 \cdot 3 = 190 = 2 \cdot 95.$$

It is clear that if $n < 95$ the average term is smaller than 2 and if $n > 95$ it is larger than 2. Thus, $n = 95$ is the unique solution.

7. Lattice points. (4,1,2,3,3,6,3,6,14,5,23)

The lattice points on the curve are (2,1), (1,2), $(-2,-1)$, $(-1,-2)$, $(4,5)$, $(5,4)$, $(-4,-5)$ and $(-5,-4)$. Obviously if (x, y) is a solution, so are (y, x), $(-x, -y)$ and $(-y, -x)$. Also, $xy > 0$, so x and y are both positive or both negative. Suppose that (x, y) is a solution, with $0 < y \le x$. Then

$$x^3 + \frac{y^4 + 79}{x} = 48y \le 48x.$$

It follows that $x^2 < 48$, and thus $|x|$ and $|y|$ in the original problem are bounded by 6. Also, one must be even and the other odd. It follows quickly that the only solutions are those listed above.

8. Missing constant term. (8,1,0,1,0,0,1,4,4,30,21)

We know immediately from the derivative that $P(x) = x^4 + 13x^3 - 6x^2 - 138x + c$ for some constant c. We show that $c = -288$. Let t and u be the other two complex roots of $P(x) = 0$. From the facts that $r + s + t + u = -13$ and $r + s = -9$ we know that $t + u = -4$. Then

$$(x - r)(x - s) = x^2 + 9x + a \quad \text{and} \quad (x - t)(x - u) = x^2 + 4x + b,$$

where $a = rs$ and $b = tu$. It follows that

$$\begin{aligned} x^4 + 13x^3 - 6x^2 - 138x + c &= (x^2 + 9x + a)(x^2 + 4x + b) \\ &= x^4 + 13x^3 + (36 + a + b)x^2 + (4a + 9b)x + ab, \end{aligned}$$

so $36 + a + b = -6$ and $4a + 9b = -138$. Solving these two linear equations for a and b we find $a = -48$ and $b = 6$. Thus $c = ab = -288$.

(If one does not assume that r and s account for two of the four zeros of P, as was intended but not explicitly stated, there is a different solution with $r = s = -9/2$ as a simple zero. Then $x + 9/2$ is a factor provided that $c = 4401/16$.)

9. Rational numbers in an interval. (10,2,1,1,1,2,0,4,1,6,42)

Suppose, on the contrary, that

$$0 < \frac{m+1}{n} < b \leq 1.$$

Then $m+1 < n$, and it would follow that

$$\frac{m}{n-1} < \frac{m+1}{n},$$

as one sees by clearing of fractions. But then

$$\frac{m}{n} < \frac{m}{n-1} < \frac{m+1}{n},$$

putting $\frac{m}{n-1}$ in (a, b), in contradiction to the minimality of n.

10. Three points in a disc. (14,0,0,0,0,0,0,0,0,6,26,24)

Draw lines parallel to the sides of the square at a spacing of $\frac{1}{5}$, partitioning the square into 25 squares of side $\frac{1}{5}$. From the fact that $51 > 25 \cdot 2$, we conclude that at least one of these squares contains at least three of the 51 points, on or interior to the square. The diagonal of this square has length $\frac{\sqrt{2}}{5} < \frac{2}{7}$ because $\frac{2}{25} < \frac{4}{49}$. Thus a circle of radius $\frac{1}{7}$ centered at the center of this square contains the square in its interior and therefore contains at least three of the 51 points.