SIXTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 16, 2002

9:00 a.m. to 12:00 noon

To the team members: These problems are meant to be fun as well as challenging. If you can work all ten of them in the allotted time, super! But if not, you are likely to have lots of good company. Partial credit will be given for significant progress or for significant partial solutions, but a thorough job on a few will usually be better than some exploratory work on many. Each problem counts 10 points.

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted.

Each team may submit one solution to each problem. Think of your solution as an essay; a logical argument which makes clear why your answer to the question is correct, or why the assertion whose proof is called for in the problem is true.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be clearly given at the top of each sheet of paper submitted.

1. Even periodic.

Recall that a function f is *even* if $f(-x) = f(x)$ for all x, and that f is *periodic* with period a means that $f(x+a) = f(x)$ for all x. Suppose that $f\colon \mathbf{R} \to \mathbf{R}$ is even, periodic with period 2, and $f(x) = x$ for $0 < x < 1$. Find $f(-\pi)$, and defend your answer.

2. An arithmetic progression.

If the pth term of an arithmetic progression is q and the qth term is p, where $p \neq q$, find the $(p+q)$th term.

3. This year's term.

For every positive integer k, define $f_1(k)$ to be the square of the sum of the digits of k (in decimal form). For $n \geq 2$, define $f_n(k) = f_1(f_{n-1}(k))$. Determine $f_{2002}(101)$.

4. A definite integral.

Let $f(x) = \int_0^x e^{-t^2} dt$. Given that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, evaluate

$$\int_0^{\infty} e^{-x^2+f(x)} dx.$$

5. Average distance 1/2.

Real numbers $a_1, a_2, \ldots, a_n$ are given, all lying in the interval $[0, 1]$. Prove that there exists a real number x in the interval such that the average of the (unsigned) distances from x to the a_i is $1/2$.

6. Solve for n.

We denote, as usual, the greatest integer not exceeding x by $\lfloor x \rfloor$. Find all integers n such that

$$\lfloor \sqrt[4]{1} \rfloor + \lfloor \sqrt[4]{2} \rfloor + \lfloor \sqrt[4]{3} \rfloor + \cdots + \lfloor \sqrt[4]{n} \rfloor = 2n.$$

7. Lattice points.

Find all lattice points (x, y) satisfying $x^4 + y^4 + 79 = 48xy$. (Lattice points are points with integer coordinates.)

8. Missing constant term.

Find the polynomial $P(x)$, given that

$$P'(x) = 4x^3 + 39x^2 - 12x - 138,$$

and that there are complex numbers r and s with $r + s = -9$ and $P(r) = P(s) = 0$.

9. Rational numbers in an interval.

Suppose that $0 \le a < b \le 1$ and that m/n is a rational fraction with minimal denominator in the open interval (a, b). That is,

$$a < \frac{m}{n} < b, \quad 0 < m < n,$$

and no rational fraction with positive denominator smaller than n lies in (a, b). Prove that $\frac{m+1}{n}$ is not in (a, b).

10. Three points in a disc.

Given a set of 51 points in a unit square, show that there is always a set of three of these points which lie interior to a circular disc of radius $\frac{1}{7}$.