

Solutions, 2000 NCS/MAA TEAM COMPETITION

1. Average velocity.

It is at $t = 3\sqrt{3}$. The total distance travelled is

$$d = \int_0^9 3t^2 dt = t^3 \Big|_0^9 = 9^3,$$

so the average velocity is $9^2 = 81$. Its velocity is 81 when $3t^2 = 81$; i.e., when $t = \sqrt{27} = 3\sqrt{3}$.

2. Smoking and heart disease.

Let the total number of people be $1000x$. Then $20x$ have heart disease, and of these, $12x$ are smokers. There are $980x$ with no heart disease, of which $196x$ are smokers. Thus $12x$ of the $208x$ smokers have heart disease, and the fraction called for is $12/208 = 3/52$.

3. Too large a rectangle.

If the rectangle does not extend above the curve, then it is contained in some rectangle inscribed in the region between the curve and the x -axis with base on the x -axis, so it suffices to show that the maximum area of such a rectangle is less than 257. It is, in fact, 256, as we show. Let $(x, 80 - x^4)$ be the upper right corner of a rectangle inscribed in the region described. Then the area of the rectangle is at most

$$A(x) = 2x(80 - x^4) = 160x - 2x^5, \quad 0 \leq x \leq \sqrt[4]{80}.$$

Since $A(0) = A(\sqrt[4]{80}) = 0$ and $A(x)$ is positive for $0 < x < \sqrt[4]{80}$, the maximum value of $A(x)$ on this closed interval occurs at an interior point, where $A'(x) = 0$. Now $A'(x) = 160 - 10x^4 = 10(16 - x^4) = 0$ only at $x = 2$ in this interval. Therefore the maximum area is $A(2) = (4)(64) = 256$. ■

Second solution, courtesy of Charles McCarthy (U.MN-TC)

It suffices to show that $257/x > -80 + x^4 > 0$ for all $x > 0$; equivalently, that

$$f(x) := 257 - 80x + x^5 > 0 \quad \text{for } x > 0.$$

If $x \geq 3$, then $-80x + x^5 \geq -80x + 3^4x \geq 3$ so $f(x) \geq 260$. If $0 < x < 3$, then $f(x) \geq 257 - 80 \cdot 3 = 17$.

4. Limit of a fraction.

The limit is $\sin 9$. If $f(x) = \int_1^x g(t)dt$ and g is continuous, then by the Fundamental Theorem of Calculus, $f'(x) = g(x)$. In particular here, if $f(x) = \int_1^x \sin(t^2)dt$, then we have $f'(x) = \sin(x^2)$. Then

$$\lim_{h \rightarrow 0} \frac{\int_1^{3+h} \sin(t^2)dt - \int_1^3 \sin(t^2)dt}{h} = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = f'(3),$$

and $f'(3) = \sin(3^2) = \sin 9$.

Second solution

If we call the numerator $F(h)$, then $F'(h) = \sin(3+h)^2$. By L'Hôpital's Rule, the limit of the fraction is

$$\frac{\lim_{h \rightarrow 0} F'(h)}{1} = \lim_{h \rightarrow 0} \frac{\sin(3+h)^2}{1} = \sin 9.$$

5. Evenly spaced roots.

The only such k is $\frac{7}{4}$. Let $f(x) = (x^2 - 1)(x^2 - 4) - k$. Since $f(-x) = f(x)$, the roots are of the form $-b, -a, a, b$, with $b > a > 0$. For these to be equally spaced we need $b - a = a - (-a)$, so $b = 3a$. Thus,

$$f(x) = (x - a)(x + a)(x - 3a)(x + 3a);$$

i.e.,

$$x^4 - 5x^2 + (4 - k) = (x^2 - a^2)(x^2 - 9a^2) = x^4 - 10a^2x^2 + 9a^4.$$

By comparing coefficients of x^2 we see that $a^2 = 1/2$, and by comparing constant terms, that $4 - k = 9a^4 = 9/4$, so that $k = 4 - 9/4 = 7/4$.

6. Final ones.

Yes. We prove by induction that for every positive integer m there is an integer n ending in 1 for which n^3 ends in m ones. With $m = 1$ we may use $1^3 = 1$. Suppose that n is an integer ending in 1 for which n^3 ends in m ones; say

$$n^3 = r \cdot 10^m + 11 \cdots 1,$$

where there are m final ones. Then if k is any positive integer we have

$$\begin{aligned} (k \cdot 10^m + n)^3 &= k^3 10^{3m} + 3k^2 10^{2m} n + 3k 10^m n^2 + n^3 \\ &= A \cdot 10^{2m} + (3kn^2 + r)10^m + 11 \cdots 1, \end{aligned}$$

where A is a positive integer and there are m ones in the final group. This number will end in $m + 1$ ones provided that $3kn^2 + r$ ends in 1. Since n ends in 1, $3kn^2$ has the same final digit as $3k$, so it suffices that $3k + r$ end in 1. For every positive integer r such a k exists, since 3 has an inverse modulo 10, ($3 \cdot 7 \equiv 1 \pmod{10}$), and the proof is complete.

7. Simplify this sum.

The sum is $\frac{n(n+1)}{2}$.

$$\begin{aligned}
 f(n) &= \sum_{m=1}^n \sum_{k=(m-1)^2+1}^{m^2} \frac{n - \lfloor \sqrt{k-1} \rfloor}{\sqrt{k} + \sqrt{k-1}} = \sum_{m=1}^n \sum_{k=(m-1)^2+1}^{m^2} \frac{n - (m-1)}{\sqrt{k} + \sqrt{k-1}} \\
 &= \sum_{m=1}^n (n - m + 1) \sum_{k=(m-1)^2+1}^{m^2} \frac{1}{\sqrt{k} + \sqrt{k-1}} \\
 &= \sum_{m=1}^n (n - m + 1) \sum_{k=(m-1)^2+1}^{m^2} (\sqrt{k} - \sqrt{k-1}) \\
 &= \sum_{m=1}^n (n - m + 1)(m - (m-1)) = \sum_{m=1}^n (n - m + 1) \\
 &= n(n+1) - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}.
 \end{aligned}$$

8. Trigonometric inequality. (Solution by Ron Rietz, Gustavus Adolphus College)

By the AM,GM inequality,

$$\begin{aligned}
 \cos x + \cos y + \sin x \sin y &\leq \cos x + \cos y + \frac{\sin^2 x + \sin^2 y}{2} \\
 &= \frac{1}{2}[2 \cos x + 2 \cos y + 1 - \cos^2 x + 1 - \cos^2 y] \\
 &= \frac{1}{2}[4 - (1 - \cos x)^2 - (1 - \cos y)^2] \leq 2.
 \end{aligned}$$

Second solution, courtesy of Charles McCarthy

$$\cos x + \cos y + \sin x \sin y = (\cos x, 1, \sin x) \cdot (1, \cos y, \sin y) \leq |(\cos x, 1, \sin x)| |(1, \cos y, \sin y)| = \sqrt{2}\sqrt{2} = 2.$$

9. Perimeter 6 and integral area.

There is just one such triangle. Its legs are $a = (5 + \sqrt{7})/3$ and $b = 2/a$, and its area is $ab/2 = 1$. Here is a proof. For a right triangle with legs a and b to have perimeter 6 and integral area it is necessary and sufficient that

$$a + b + \sqrt{a^2 + b^2} = 6 \tag{1}$$

and

$$ab = 2n, \tag{2}$$

where n is a positive integer. From $\sqrt{a^2 + b^2} = 6 - (a + b)$ we have

$$a^2 + b^2 = 36 - 12(a + b) + a^2 + 2ab + b^2,$$

which simplifies to

$$3(a + b) = 9 + n. \tag{3}$$

There are various ways to obtain an upper bound on n . E.g., $3 < a + b < 4$ because the two legs must account for more than half and less than $2/3$ of the perimeter. Then (3) yields $9 < 9 + n < 12$, so $n = 1$ or 2 . But if $n = 2$ then $a + b = \frac{11}{3}$, and substituting $b = \frac{11}{3} - a$ into (1) yields only nonreal values for a . Thus $n = 1$ is the only possibility. Then from (3) and (2) we have $a + b = 10/3$ and $ab = 2$, which leads to $a + \frac{2}{a} = 10/3$. The only positive root is $a = (5 + \sqrt{7})/3$, which together with $b = 2/a$ give the asserted solution.

10. Bigger than 2000 for large n . Solution by Macalester Team 4.

Any integer $n > e^{2000}$ suffices. For,

$$\sum_{k=n}^{n^2} \frac{1}{k} = \int_n^{n^2+1} \frac{1}{[x]} dx > \int_n^{n^2} \frac{1}{x} dx = \ln n,$$

and $\ln n > 2000$ when $n > e^{2000}$.

Second solution, by Carleton A team

We claim that $n = 2^{4000}$ suffices. With this value of n , (omitting the term $\frac{1}{n}$),

$$\begin{aligned} \sum_{k=n}^{n^2} \frac{1}{k} &> \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right) + \left(\frac{1}{2n+1} + \cdots + \frac{1}{4n} \right) + \cdots + \left(\frac{1}{2^{3999}n+1} + \cdots + \frac{1}{2^{4000}n} \right) \\ &> \frac{n}{2n} + \frac{2n}{4n} + \cdots + \frac{2^{3999}n}{2^{4000}n} \\ &= \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} = \frac{4000}{2} = 2000. \end{aligned}$$