Solutions, 1999 NCS/MAA TEAM COMPETITION

1. Mind reading.

Let the three numbers be a, b and c. Then the result of the calculation is

$$((5a+7)\cdot 2+b)\cdot 10+c,$$

which upon expansion may be written

$$100a+10b+c+140.$$

If you subtract 140 from this result, you have the three digit number with digits a, b and c. Thus

(a) $401-140=261$, so the numbers she rolled were 2, 6 and 1, in that order. (One still needs to show this solution is unique. To do

so, look at the equation $100a+10b+c=261$ modulo 10 to get $c=1$, subsitute this back in, divide by 10, etc.)

(b) Subtract 140 from the announced result, and the three digits
of the resulting number are the numbers on the dice, in order.

2. Function iteration.

The answer is $f_{2000}(1999)=\frac{1998}{1999}$. We have

$$f_2(x)=\frac{1}{1-\frac{1}{1-x}}=\frac{x-1}{x},$$

and

$$f_3(x)=\frac{1}{1-(\frac{x-1}{x})}=x.$$

Then $f_4(x)=f_1(x)$, and for each n, $f_{n+3}(x)=f_n(x)$. Since $2000\equiv 2 \pmod 3$, we have $f_{2000}(x)=f_2(x)=\frac{x-1}{x}$, and $f_{2000}(1999)=\frac{1998}{1999}$.

3. Unique factorization.

No, it is not unique. We show that $(x+4)(x+5)=x^2+3x+2$ is a different factorization. Direct computation shows that this is a valid factorization. Now, the only units in the ring $R[x]$ are those in R, namely 1 and 5: $1\cdot 1=1$ and $5\cdot 5=1$. Thus the only associates of $(x+1)$ are itself and $5(x+1)=5x+5$, and of $(x+2)$, itself and $5(x+2)=5x+4$. Thus $(x+4)(x+5)$ is in fact a different factorization of x^2+3x+2.

4. An integral.

We show that the value of the integral is $-\frac{1}{3}+\frac{1}{2}\sqrt{2}+\frac{1}{6}\sqrt{3}$. From the definition of the floor function,

$$\frac{1}{\lfloor x^2\rfloor}=\begin{cases}\dfrac{1}{1}, & 1\le x<\sqrt{2}\\ \dfrac{1}{2}, & \sqrt{2}\le x<\sqrt{3}\\ \dfrac{1}{3}, & \sqrt{3}\le x<2.\end{cases}$$

Then

$$\begin{aligned}\int_1^2 \frac{1}{\lfloor x^2\rfloor}dx &= 1(\sqrt{2}-1)+\frac{1}{2}(\sqrt{3}-\sqrt{2})+\frac{1}{3}(2-\sqrt{3})\\ &= -\frac{1}{3}+\frac{1}{2}\sqrt{2}+\frac{1}{6}\sqrt{3}.\end{aligned}$$

5. Every non-constant function?

Suppose on the contrary that for every x and h we have $|f(x+h)-f(x)|\le |h|^{3/2}$. Then

$$\left|\frac{f(x+h)-f(x)}{h}\right|\le |h|^{\frac{1}{2}}.$$

It follows at once that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=0$$

for every x, and therefore that f is constant. ∎

Second solution

Here is a direct proof. Let a and b be points with $a<b$ and $f(a)\ne f(b)$, and let $c=|f(b)-f(a)|$. Given any positive integer n, partition the interval $[a,b]$ into n subintervals, each of length $(b-a)/n$ by the points $a=a_0<a_1<\cdots<a_n=b$, and write

$$c=|f(b)-f(a)|=|\sum_{k=1}^{n} f(a_k)-f(a_{k-1})|\le\sum_{k=1}^{n}|f(a_k)-f(a_{k-1})|,$$

so for some k we have

$$\begin{aligned}|f(a_k)-f(a_{k-1})|&\ge\frac{c}{n}=\frac{c}{n}\Big(\frac{n(a_k-a_{k-1})}{b-a}\Big)^{\frac{3}{2}}\\ &=\frac{c\sqrt{n}}{(b-a)^{\frac{3}{2}}}(a_k-a_{k-1})^{\frac{3}{2}}>(a_k-a_{k-1})^{\frac{3}{2}}\end{aligned}$$

when

$$\sqrt{n}>\frac{(b-a)^{\frac{3}{2}}}{c}.$$

6. A double inequality.

We begin with the first inequality. Since $2a < b + c$, this is equivalent to

$$ab + ac - 2a^2 < bc - a^2;$$

i.e., to

$$ab - a^2 < bc - ac;$$

i.e., to

$$a(b - a) < c(b - a). \tag{1}$$

Since $a < c$ and $b - a > 0$, (1) is true, so the desired inequality is likewise.

Note that because of the symmetry in b and c in the problem, we may assume without loss of generality that $b \leq c$. The second inequality is then equivalent to

$$bc - a^2 < b^2 + bc - 2ab;$$

i.e., to

$$0 < a^2 - 2ab + b^2 = (a - b)^2,$$

which is true because $a \neq b$. Therefore the second inequality holds as well.

7. A pair of equations.

There are no solutions. For, these equations imply that $a = 1999^{z/x}$ and $b = 1999^{z/y}$, with $z/x + z/y = 1$, so that $ab = 1999$. Since a and b are to be positive integers, and 1999 is prime, one of a and b must be 1; say, $a = 1$. But since $z \neq 0$, then $a = 1999^{z/x}$ is impossible.

8. Least upper bound.

The least upper bound is 1/2. There are several ways to find it. One way is to put $z = 3e^{i\theta}$, express $|z^4 - 5z^2 + 6| = |z^2 - 3||z^2 - 2|$ as a function of θ and use calculus to find its maximum value. Here is a shorter solution: By the triangle inequality we have

$$\begin{aligned} |z^4 - 5z^2 + 6| &= |z^2 - 3||z^2 - 2| \\ &\geq \Big||z^2| - 3\Big|\Big||z^2| - 2\Big| \\ &= (9 - 3)(9 - 2) = 42. \end{aligned}$$

At $z = 3$ we have $|z^4 - 5z^2 + 6| = 42$, so this is its minimum value. Hence the desired least upper bound is $21/42 = 1/2$.

9. Sum the series.

The sum is $e^{\cos 3}\cos(\sin 3)$. We show more generally that

$$C(\theta) := \sum_{n=0}^{\infty} \frac{\cos n\theta}{n!} = e^{\cos\theta}\cos(\sin\theta).$$

The key is to look at this series in tandem with the series

$$S(\theta) = \sum_{n=0}^{\infty} \frac{\sin n\theta}{n!}.$$

Then, since these series are absolutely convergent,

$$\begin{aligned} C(\theta) + iS(\theta) &= 1 + (\cos\theta + i\sin\theta) + \frac{\cos 2\theta + i\sin 2\theta}{2!} + \frac{\cos 3\theta + i\sin 3\theta}{3!} + \cdots \\ &= 1 + e^{i\theta} + \frac{e^{2i\theta}}{2!} + \frac{e^{3i\theta}}{3!} + \cdots \\ &= e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} \\ &= e^{\cos\theta}(\cos(\sin\theta) + i\sin(\sin\theta)), \end{aligned}$$

so

$$C(\theta) = e^{\cos\theta}\cos(\sin\theta) \quad \text{and} \quad S(\theta) = e^{\cos\theta}\sin(\sin\theta).$$

10. A nonnegative function.

If we put $q(x) = p(x) - p''(x)$, then the given condition can be written $q(x) - q'(x) \geq 0$. We show first that this implies $q(x) \geq 0$ for all x, and then (as we'll see) it is an easy step to show that $p(x) \geq 0$ for all x.

That $q(x) - q'(x) \geq 0$ implies that $q(x) - q'(x)$, and therefore $q(x)$ itself, has even degree and positive leading coefficient. Thus $q(x)$ has an absolute minimum at some point, say at b. Since $q'(b) = 0$ we have

$$q(x) \geq q(b) \geq q'(b) \geq 0$$

for all x, as claimed.

This implies that $p(x) - p''(x) = q(x)$ is a polynomial of even degree with leading coefficient positive. Then $p(x)$ itself is a polynomial of even degree with leading coefficient positive. Therefore $p(x)$ has an absolute minimum value, say at $x = a$. At a minimum point, the second derivative satisfies $p''(a) \geq 0$. Then for all x we have $p(x) \geq p(a) \geq p''(a) \geq 0$. ∎