Solutions, 1998 NCS/MAA TEAM COMPETITION

1. Matching pennies.

Well, let x be the number of tosses. Adolph won 13 of them, so Bertha won the other $x-13$. Then Bertha's net gain was $(x-13)-13$ pennies. Thus, $x-26=8$, and $x=34$.

ALTERNATE SOLUTION

Since Adolph won on 13 of the tosses, Bertha had to win on 13 others to break even, and on 8 more to be 8 ahead. Thus Bertha won on 21 of the tosses, and there were $13+21=34$ tosses altogether.

2. Base *b* quadratic.

The answer is $n = 36_b$. Since $(x-9)(x-5) = x^2 - 11_b x + n$, we know that $9+5 = (11)_b = b+1$, so $b = (13)_{10}$. Then, since $(9)(5) = 3b+6$, we have $n = 36_b$.

3. Hyperbolics.

The answer is $16\sqrt{2}$. From the definitions of sinh and cosh we have

$$2 = \sinh 2x + \cosh 2x = e^{2x}.$$

Then

$$\sinh 9x + \cosh 9x = e^{9x} = (e^{2x})^{9/2} = 2^{9/2} = 16\sqrt{2}.$$

4. A relatively prime sequence.

Note first that all a_n are odd (by an easy induction). Assume that $m < n$. Then

$$a_{m+1} = a_m^2 - 2 \equiv -2 \pmod{a_m}$$
$$a_{m+2} \equiv (-2)^2 - 2 \equiv 2 \pmod{a_m},$$

and by induction, for every $k \geq 2$, $a_{m+k} \equiv 2 \pmod{a_m}$. Thus $a_n = ra_m + 2$ or $ra_m - 2$ for some integer r, and therefore every common factor of a_m and a_n is a divisor of 2. Since both a_m and a_n are odd, they are relatively prime. ∎

5. Sum the series.

We will show that the sum is $\ln 6$. Using well-known properties of the logarithm, we write the partial sum up to N (with $N > 4$)

$$\begin{aligned}
&\sum_{n=2}^{N}[3\ln n - 2\ln(n-1) - \ln(n+2)] \\
&\quad = 3[\qquad \ln 2 + \ln 3 + \ln 4 + \cdots + \ln(N-1) + \ln N] \\
&\quad - 2[\ln 1 + \ln 2 + \ln 3 + \ln 4 + \cdots + \ln(N-1)] \\
&\quad - [\qquad \ln 4 + \cdots + \ln(N-1) + \ln N + \ln(N+1) + \ln(N+2)].
\end{aligned} \tag{1}$$

The terms

$$\ln 4 + \cdots + \ln(N-1)$$

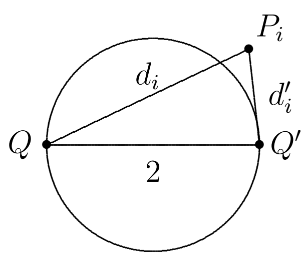
in (1) cancel out, leaving

$$\begin{aligned}
\sum_{n=2}^{N} \ln \frac{n^3}{(n+2)(n-1)^2} &= \ln 2 + \ln 3 + 2\ln N - \ln(N+1) - \ln(N+2) \\
&= \ln 6 + \ln \frac{N^2}{(N+1)(N+2)},
\end{aligned}$$

which approaches $\ln 6$ as $N \to \infty$.

6. A circle and some points.

Let the chosen points be $P_1, P_2, \ldots, P_{1998}$. Choose any two diametrically opposite points Q and Q' on the circle, subject only to the restriction that not all P_i lie on the segment QQ',



and let d_i and d_i' be the distances from Q and Q', resp. to P_i, $1 \le i \le 1998$. By the triangle inequality, $d_i + d_i' \ge 2$, with equality if and only if P_i is on the segment QQ'. Since some of the P_i are not on this segment we have $\sum d_i + \sum d_i' > 2 \cdot 1998$, so at least one of $\sum d_i$ and $\sum d_i'$ is greater than 1998. ∎

7. Equal integrals.

For simplicity we translate the origin to the midpoint of the interval (a, b). Let $h = (b-a)/2$, $F(x) = f(x + (a+b)/2)$ and $G(x) = g(x + (a+b)/2)$. Then

$$F(-h) = f(a) = g(a) = G(-h),$$

$$F(0) = f\left(\frac{a+b}{2}\right) = g\left(\frac{a+b}{2}\right) = G(0)$$

and

$$F(h) = f(b) = g(b) = G(h).$$

Also,

$$\int_a^b f(x)dx = \int_{-h}^h F(x)dx$$

and

$$\int_a^b g(x)dx = \int_{-h}^h G(x)dx,$$

so it suffices to show that

$$\int_{-h}^h F(x)dx = \int_{-h}^h G(x)dx.$$

Let $P(x) = F(x) - G(x)$. Then $P(x)$ is a polynomial of degree 3 with zeros at $-h, 0$ and h, so for some constant c,

$$P(x) = cx(x-h)(x+h) = cx(x^2 - h^2). \tag{1}$$

¿From (1) we see that P is an odd function, and therefore

$$\int_{-h}^h P(x)dx = 0,$$

q.e.d.

Note: This fact may also be seen as a consequence of the fact that in Simpson's Rule the error is bounded by a multiple of the fourth derivative. Simpson's Rule with $n = 2$ approximates

$$\int_a^b g(x)dx \quad \text{by} \quad \int_a^b f(x)dx,$$

where $f(x)$ is a quadratic polynomial fitted to $g(x)$ at $a, (a+b)/2$ and b. The fourth derivative of $g(x)$ is, of course, identically zero.

8. Reciprocal inequality.

We shall repeatedly use the fact that for all real numbers u and v,

$$u^2 + v^2 \geq 2uv, \tag{1}$$

which follows at once from $(u-v)^2 \geq 0$. We apply (1) to obtain

$$a^4 + b^4 \geq 2a^2b^2, \quad \text{and} \quad c^4 + d^4 \geq 2c^2d^2.$$

Thus

$$a^4 + b^4 + c^4 + d^4 \geq 2(a^2b^2 + c^2d^2). \tag{2}$$

Another application of (1) tells us that

$$a^2b^2 + c^2d^2 \geq 2(abcd), \tag{3}$$

and from (2) and (3) we conclude that

$$a^4 + b^4 + c^4 + d^4 \geq 4abcd. \tag{4}$$

Applying (4) to $1/a$, $1/b$, $1/c$ and $1/d$ we have

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} \geq \frac{4}{abcd}. \tag{5}$$

Thus we have

$$1 \geq a^4 + b^4 + c^4 + d^4 \geq 4abcd,$$

so that

$$abcd \leq \frac{1}{4}$$

and therefore

$$\frac{1}{abcd} \geq 4.$$

Then from (5) it follows that

$$\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4} + \frac{1}{d^4} \geq 16,$$

as desired.

9. Order of an element.

The order of y is 31. From $xyx^{-1} = y^2$ we have $xy = y^2x$, and a key observation is that for every positive integer n,

$$xy^n = y^{2n}x, \tag{1}$$

as we show by induction. We already know it for $n = 1$, and if $xy^k = y^{2k}x$, then

$$xy^{k+1} = y^{2k}xy = y^{2k}y^2x = y^{2k+2}x,$$

so we have (1) by induction. Since $x^5 = e$, we have

$$\begin{aligned} y &= x^5yx^{-5} \\ &= x^4(y^2x)x^{-5} \\ &= x^4y^2x^{-4} \\ &= x^3y^4x^{-3} \\ &= x^2y^8x^{-2} \\ &= xy^{16}x^{-1} \\ &= y^{32}; \end{aligned}$$

i.e., $y = y^{32}$, so $y^{31} = e$. Since 31 is prime, it is the order of y.

10. Not a fifth power.

WLOG we may assume that $\epsilon_{60} = 1$. Let $r = 60^{\frac{1}{5}(60^{60})}$. We show that

$$\text{if } \epsilon_{59} = -1, \quad \text{then } (r-1)^5 < N < r^5, \tag{1}$$

and

$$\text{if } \epsilon_{59} = 1, \quad \text{then } r^5 < N < (r+1)^5. \tag{2}$$

For (1) we note first that

$$60^{60} = (59+1)^{60} > 59^{60} + 60 \cdot 59^{59} > 2 \cdot 59^{60}, \tag{3}$$

so that (using (3) in the second step following)

$$r^3 = 60^{\frac{3}{5}(60^{60})} > 60^{\frac{6}{5}(59^{60})} > 59^{(59^{60})} > 59^{(59^{59}+1)} = 59 \cdot 59^{(59^{59})}. \tag{4}$$

Then

$$\begin{aligned}(r-1)^5 &= r^5 - 5r^3(r-2) - 5r(2r-1) - 1 \\ &< r^5 - 5r^3(r-2) \\ &< r^5 - 5r^3 \\ &< r^5 - r^3,\end{aligned}$$

and from (4) we have

$$\begin{aligned}r^5 - r^3 &< r^5 - 59 \cdot 59^{59^{59}} \\ &< 60^{60^{60}} + \sum_{k=1}^{59} (-1) k^{k^k} \\ &\leq N \\ &< r^5 - 59^{(59^{59})} + 58 \cdot 58^{(58^{58})} \\ &= r^5 - 59^{(59^{59})} + 58^{(58^{58}+1)} \\ &< r^5 - 59^{(59^{59})} + 58^{(58^{59})} \\ &< r^5.\end{aligned}$$

A similar argument proves (2).