Solutions, 1997 NCS/MAA TEAM COMPETITION

1. Social optimism.

The answer is, $SO_2 = -20$. For,

$$\begin{aligned} 100 &= a + b + c \\ &= \left(a + \frac{b}{2}\right) + \left(\frac{b}{2} + c\right) \\ &= 40 + \left(\frac{b}{2} + c\right), \end{aligned}$$

so $(b/2) + c = 60$. Then

$$\begin{aligned} a - c &= \left(a + \frac{b}{2}\right) - \left(\frac{b}{2} + c\right) \\ &= 40 - 60 = -20. \end{aligned}$$

2. Sums of cubes.

Let n be any integer. Then

$$(n-1)^3 + n^3 + (n+1)^3 = 3n^3 + 6n = 3n(n^2 + 2).$$

If $n \equiv 0 \pmod 3$ then $3n$ is a multiple of 9. If $n \equiv \pm 1 \pmod 3$ then $n^2 + 2$ is a multiple of 3. In either case, $3n(n^2 + 2)$ is a multiple of 9.

3. A nonlinear equation.

One sees by inspection that $f(0) = n$. We show that $f(x) = n$ has no other solution. Since $\cos x\sqrt{k} \le 1$ for each k, $f(x) = n$ requires $\cos x\sqrt{k} = 1$ for each k. In particular, $\cos x = 1$ and $\cos x\sqrt{2} = 1$. Then there are integers r and s such that $x = 2\pi r$ and $x\sqrt{2} = 2\pi s$, and hence $s = r\sqrt{2}$. Since, as is well known, $\sqrt{2}$ is irrational, this is impossible unless $r = s = 0$.

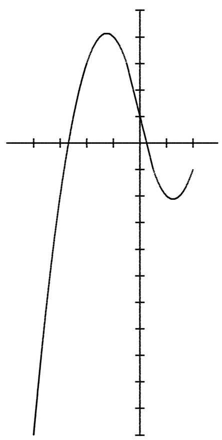
4. Max and min.

The maximum value is $33/8 = 4.125$, at $x = -5/4$, and the minimum value is -11, at $x = -4$. To see this, let

$$f(x) = 2x|x| - 5x + 1 = \begin{cases} 2x^2 - 5x+1, & \text{if } x \geq 0; \\ -2x^2 - 5x+1, & \text{if } x < 0. \end{cases}$$

Then

$$f'(x) = \begin{cases} 4x - 5, & \text{if } x > 0; \\ -4x - 5, & \text{if } x < 0. \end{cases}$$



For $x \geq 0$ this is a parabola opening upward from $(5/4, -17/8)$ and for $x < 0$, a parabola opening downward from $(-5/4, 33/8)$. The range $|x+1| \leq 3$ is equivalent to $-3 \leq x + 1 \leq 3$; i.e., $-4 \leq x \leq 2$. The function increases for $-4 \leq x \leq -5/4$, decreases for $-5/4 \leq x \leq 5/4$, and increases for $5/4 \leq x \leq 2$. Candidates for local extrema are at $x = -4, -5/4, 5/4$ and 2, where $f(x)$ has values $-11, 33/8 = 4.125, -17/8 = -2.125$ and -1, respectively. Thus the maximum value is $33/8$, at $x = -5/4$, and the minimum is -11, at $x = -4$.

5. Polynomial evaluation.

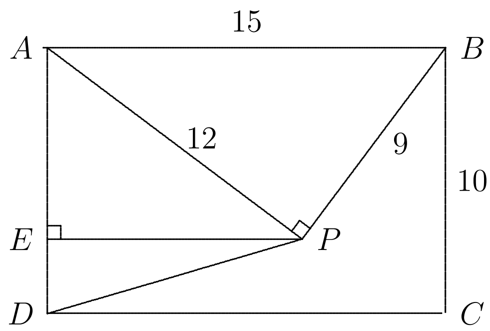
It is -1, for $4x^2 - 4x + 1 = (2x-1)^2 = 1997$, so $4x^2 = 4x + 1996$. Thus $4x^3 = 4x^2 + 1996x$, whence

$$\begin{aligned} 4x^3 - 2000x - 1997 &= 4x^2 + 1996x - 2000x - 1997 \\ &= 4x^2 - 4x - 1997 \\ &= 4x^2 - 4x + 1 - 1998 \\ &= 1997 - 1998 \\ &= -1. \end{aligned}$$

Thus $(4x^3 - 2000x - 1997)^{2003} = (-1)^{2003} = -1$.

(Alternatively, one may calculate $4x^3 - 2000x - 1997 = -1$ by brute force.)

6. An isosceles triangle.



Note first that triangle ABP is a (9,12,15) right triangle. Hence, if E is the foot of the perpendicular from P to AD, triangle PEA is similar to triangle ABP, with sides 4/5 as long. Consequently, we find the lengths of AE, EP and ED to be 36/5, 48/5 and 14/5, respectively. Then from the Pythagorean Theorem, the length of DP is

$$\frac{1}{5}\sqrt{14^2 + 48^2} = 10,$$

showing that triangle ADP is isosceles. ∎

7. Pick an integer.

Short solution: There are only k residue classes mod k, so among $k+1$ integer choices, some two are in the same residue class.

(Slightly expanded version) If the integers chosen are $x_1, x_2, \ldots, x_{k+1}$, let $y_i = x_{k+1} - x_i$ reduced mod k, for $i = 1, 2, \ldots, k$. Thus each $y_i \in \{0, 1, \ldots, k-1\}$. If some $y_i = 0$ we are done. If not, then the k integers $y_1, y_2, \ldots, y_k$ lie in the $(k-1)$-element set $\{1, 2, \ldots, k-1\}$, so by the pigeon-hole principle $y_i = y_j$ for some $i \neq j$, and in this case,

$$x_j - x_i = (x_{k+1} - x_i) - (x_{k+1} - x_j) \equiv 0 \pmod{k}.$$

8. f(x)=f(x+1)?

If $d = 1$ we may take $x = 0$. Assume then that $d > 1$, and let $g(x) = f(x+1) - f(x)$ for $0 \leq x \leq d-1$. We need to show that $g(x) = 0$ for some x in $[0, d-1]$. Now,

$$g(0) + g(1) + \cdots + g(d-1) = f(d) - f(0) = 0. \tag{1}$$

If $g(k) = 0$ for some k in $\{0, 1, \ldots, d-1\}$, we are done. If not,

it follows from (1) that there are r and s in $\{0, 1, \ldots, d-1\}$ such that $g(r) < 0$ and $g(s) > 0$. Then by the Intermediate Value Theorem, there is a number x

between r and s such that $g(x) = 0$, q.e.d.

9. No real roots.

The condition is $a \neq 2$. We prove this by proving that (1) has a real root if, and only if, $a = 2$.

Suppose first that (1) has a real root r. Then separating (1) into real and imaginary parts we have

$$(r^2 + ar + 1) + (-r^2 + r + a)i = 0,$$

so that

$$a = r^2 - r$$

and

$$r^2 + ar + 1 = r^2 + (r^2 - r)r + 1 = 0;$$

i.e.

$$r^3 + 1 = 0.$$

Since r is real, we have $r = -1$ and $a = 2$.

Conversely, if $a = 2$, one verifies easily that (1) has the real root $x = -1$.

10. A term divisible by 1997?

Modulo r, there are at most r^3 different triples $(a_{k+1}, a_{k+2}, a_{k+3})$, so the sequence must eventually be periodic mod r. But since the recursion is reversible ($a_n = a_{n+3} - a_{n+1}a_{n+2}$), it

must be truly periodic mod r and hence eventually return to its initial state. At that point, when $a_{s+1} \equiv a_{s+2} \equiv a_{s+3} \equiv 1 \pmod{r}$, we have $a_s = a_{s+3} - a_{s+1}a_{s+2} \equiv 0 \pmod{r}$.